

AN INTERESTING EXAMPLE OF A COMPACT NON-C-ANALYTIC REAL SUBVARIETY OF \mathbb{R}^3

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ABSTRACT. The purpose of this short note is to provide an interesting new example exhibiting some of the pathological properties of real-analytic subvarieties. We construct a compact irreducible real-analytic subvariety S of \mathbb{R}^3 of pure dimension two such that 1) the only a real-analytic function is defined in a neighbourhood of S and vanishing on S is the zero function, 2) the singular set of S is not a subvariety of S , nor is it contained in any one-dimensional subvariety of S , 3) the variety S contains a proper subvariety of dimension two. The example shows how a badly behaved part of a subvariety can be hidden via a second well behaved component to create a subvariety of a larger set.

A closed set $S \subset \mathbb{R}^n$ is a *real-analytic subvariety* if for every $p \in S$, there exist real-analytic functions ρ_1, \dots, ρ_k defined in a neighbourhood U of p , such that $S \cap U$ is equal to the set where all ρ_1, \dots, ρ_k vanish. A complex subvariety is precisely the same notion in \mathbb{C}^n , with real-analytic replaced by holomorphic (complex analytic). See [2, 3] for more information.

Given a real-analytic function $\rho(x_1, \dots, x_n)$ defined near p , consider $\mathbb{R}^n \subset \mathbb{C}^n$, and then consider x_1, \dots, x_n as complex variables. The power series still converges in some small neighbourhood of $p \in \mathbb{C}^n$. This process is called *complexification*. Starting with a real-analytic subvariety near a point p , complexify near p to obtain a complex subvariety of a neighbourhood of $p \in \mathbb{C}^n$ whose trace on \mathbb{R}^n is the real subvariety. Despite this connection, real-analytic subvarieties have very pathological properties not present in the complex world.

The basic properties of real-subvarieties including examples of the pathologies explored in this note have been known at least since the work of Cartan [1], although there seem to be only a few examples given in the literature. This note is hoped to improve the situation. The motivation for this work is to present a new and somewhat different example subvariety, one where the pathology can be easily visualised. In particular, the author's motivating question was how can a badly behaved part of a subvariety be hidden to extend the subvariety to a larger set via a seemingly unrelated other subvariety with a very different geometry.

Let us start with the notion of irreducibility. A real-analytic subvariety $X \subset \mathbb{R}^n$ is *irreducible* if whenever we write $X = X_1 \cup X_2$ for two subvarieties X_1 and X_2 of \mathbb{R}^3 , then either $X_1 = X$ or $X_2 = X$. This notion is subtle. We will construct a set that is a union of two subvarieties, one of which is not a subvariety of \mathbb{R}^3 but of a strictly smaller domain, and the union is irreducible as a subvariety of \mathbb{R}^3 .

Let us proceed in steps. Start with the sphere $z^2 = 1 - x^2 - y^2$, thinking of z^2 as a “graph”. Pinch the sphere along the y -axis by multiplying by x^2 to obtain the subvariety S_1 given by

$$z^2 = (1 - x^2 - y^2)x^2. \quad (1)$$

The picture is the left hand side of Figure 1.

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The subvariety S_1 is irreducible, and it contains the y -axis as a subvariety. This subvariety is already somewhat pathological. It has components of different dimensions, and the regular points of dimension two are not topologically connected.

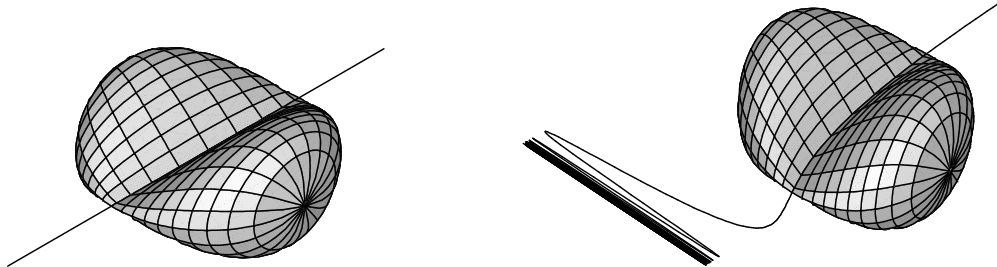


FIGURE 1. The sets S_1 (left) and S_2 (right).

We restrict our attention to the set where $-2 < y < 2$. We make a change of coordinates keeping y and z fixed, but sending x to $x + \sin\left(\frac{1}{y+2}\right)$. The equation becomes

$$z^2 = \left(1 - \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2 - y^2\right) \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2. \quad (2)$$

We call this set S_2 , restricted to $-2 < y < 2$ to make it bounded. What we have done is made the y axis appear like the graph of $\sin\left(\frac{1}{y+2}\right)$. The picture of the result is the right hand side of Figure 1. The set S_2 is a real-analytic subvariety of the set $\{-2 < y < 2\}$, and it becomes badly behaved as we approach $y = -2$.

Proposition 1. *Let $\Omega \subset \mathbb{R}^3$ be a connected neighbourhood of the closure $\overline{S_2}$ and $r: \Omega \rightarrow \mathbb{R}$ a real-analytic function vanishing on S_2 . Then r is identically zero.*

It would be very easy to prove that r has to vanish on the xy -plane via essentially just staring at the picture and recalling basic properties of real-analytic functions. However, to prove that r vanishes everywhere, we need to complexify r . The result is not just because the 1-dimensional component wiggles around, it is because the complexification of the 2-dimensional part gets dragged along the 1-dimensional component.

Proof. We consider $\mathbb{R}^3 \subset \mathbb{C}^3$. Then S_2 is also a subset of \mathbb{C}^3 . Treating (x, y, z) as complex variables, let us call \mathcal{S} the complex subvariety of $\{y \neq -2\}$ set defined by (2). The subvariety \mathcal{S} is locally irreducible at $(0, -1, 0)$. Indeed, think of z^2 as a graph over (x, y) , and so there can at most be “two sheets” in \mathcal{S} , for the two different square roots of the right hand side of (2). We can clearly move from one root to the other in an arbitrarily small neighbourhood of $(0, -1, 0)$.

Complexify r to obtain a holomorphic function \tilde{r} of a neighbourhood U of $\overline{S_2}$ in \mathbb{C}^3 . As \tilde{r} vanishes on $\overline{S_2}$, then by irreducibility of \mathcal{S} at $(0, -1, 0)$, there exists a neighbourhood W of $(0, -1, 0)$ in \mathbb{C}^3 , such that \tilde{r} vanishes on $W \cap \mathcal{S}$.

Fix $z = ia$ for a small real a . Let X_a be the set defined by

$$-a^2 = \left(1 - \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2 - y^2\right) \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2. \quad (3)$$

for real x and y with $-2 < y < -1$. The set X_a is a connected smooth real-analytic curve, which is a subset of \mathcal{S} . If a is small enough, $X_a \subset \mathcal{S} \cap U$ and $X_a \cap W$ is nonempty. The function \tilde{r} then vanishes on $X_a \cap W$, an open set of X_a , and hence on all of X_a .

Next fix a small real x . The equation (3) is true for an infinite sequence of y approaching $y = -2$ from above. Therefore, the holomorphic function of one complex variable $y \mapsto \tilde{r}(x, y, ia)$ defined in a neighbourhood of the origin vanishes identically. As this was true for all small enough x and a , it is true for small enough complex x and a , and \tilde{r} vanishes in a neighbourhood of $(0, -2, 0)$ in \mathbb{C}^3 . By analytic continuation \tilde{r} is identically zero. \square

To visualize how bad the complexification is as we approach $y = -2$, consider the set in the space $(x, y, a) \in \mathbb{R}^3$ given by (3). Looking at the set where $a \geq 0$, we have a “valley” whose bottom is the graph $x = \sin(\frac{1}{y+2})$ with increasingly steep sides. See Figure 2.

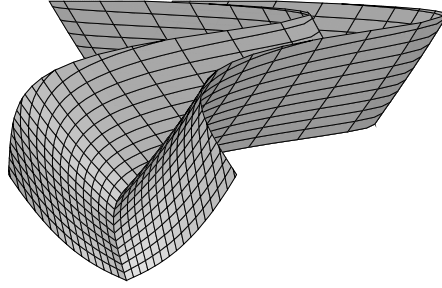


FIGURE 2. The trace of the complexification of S_2 in (x, y, a) -space for $a \geq 0$ and $y < -1$, approaching $y = -2$.

Let us “hide” the wild behavior near $y = -2$ and construct the purely 2-dimensional subvariety S_3 via

$$S_3 = S_2 \cup \{z = 0\}. \quad (4)$$

The picture is the left hand side of Figure 3. Suppose $\Omega \subset \mathbb{R}^3$ is a connected neighbourhood of S_3 and $r: \Omega \rightarrow \mathbb{R}$ a real-analytic function such that $r = 0$ on S_3 . The set Ω is also a neighbourhood of $\overline{S_2}$ and $r = 0$ on $\overline{S_2}$. By the proposition, $r \equiv 0$. In the terminology of real-analytic varieties, S_3 is not \mathbb{C} -analytic.

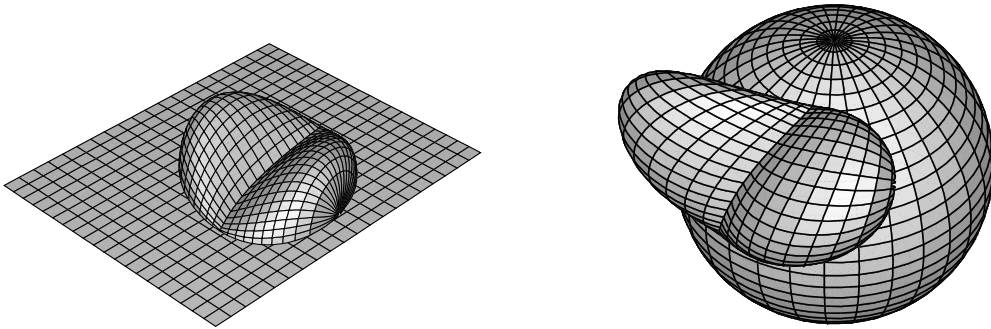


FIGURE 3. The sets S_3 (left) and S_4 (right).

The singular set of S_3 is the set

$$\begin{aligned} z &= 0, \\ -1 &\leq y \leq 1, \\ 0 &= \left(1 - \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2 - y^2\right) \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2. \end{aligned} \quad (5)$$

This singular set clearly is not a subvariety. In particular, it contains the set

$$I = \left\{ (x, y, z) : x = \sin\left(\frac{1}{y+2}\right) \quad \text{and} \quad -1 \leq y \leq 1 \right\}, \quad (6)$$

and I cannot be contained in any subvariety of S_3 of dimension 1. Any such subvariety would have to contain the entire set $x = \sin\left(\frac{1}{y+2}\right)$ for all $y > -2$, and it cannot possibly be a subvariety at points where $y = -2$.

The subvariety S_3 is irreducible. It contains a proper subvariety of dimension 2, namely the xy -plane. Any subvariety S' that contains any open set of the regular points must contain I . Indeed, if S' contains an open set of the xy -plane it must contain whole xy -plane. If S' contains an open set of one of the smooth submanifolds outside of the xy -plane then I is in the closure of this submanifold and hence in S' . Any subvariety that contains I must contain the entire xy -plane.

We have demonstrated a subvariety with all the required properties but not a compact one. We make the subvariety compact by mapping the plane onto the sphere using spherical coordinates. For the picture on the right hand side of Figure 3 we used the map

$$(x, y, z) \mapsto \left((z+1) \sin(1+y/2) \cos(x), (z+1) \sin(1+y/2) \sin(x), (z+1) \cos(1+y/2) \right). \quad (7)$$

For $-\pi < x < \pi$, $0 < (1+y/2) < \pi$, and $z+1 > 0$, the mapping is a real-analytic diffeomorphism and we obtain the compact subvariety S_4 by taking the closure, which will fill in the missing meridian on the far side of the sphere.

This subvariety clearly has all the properties mentioned in the abstract; it is compact and inherits the rest of the properties from S_3 .

The construction is easy to modify to show further strange behaviors. For example, if we start with S_3 , but rescale the z variable we obtain another irreducible subvariety that shares with S_3 a 2-dimensional component as a proper subvariety.

REFERENCES

- [1] Henri Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France **85** (1957), 77–99 (French). MR0094830
- [2] Francesco Gueraldo, Patrizia Macrì, and Alessandro Tancredi, *Topics on real analytic spaces*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1986. MR1013362
- [3] Hassler Whitney, *Complex analytic varieties*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972. MR0387634

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